What is Locking?

Enhanced assumed strain formulation in 2D (Simo & Rifai 1990)

B-bar formulation (Hughes 1980)

Reduced integration and hourglass stabilization (Belytschko & Bachrach 1986)

Practical importance of locking-free finite elements
What is Locking?

First evidences: 1960’s - solid elements for nearly incompressible materials

- propellant materials had nearly incompressible properties ($\nu \approx 0.5$)
- standard displacement FEM gives poor result for $\nu > 0.4$
- standard FDM also locked locking is triggered by underlying BVP
- naive mixed methods (displacement/pressure) worked satisfactory
- could be extended to elastic and thermo-viscoelastic materials

Relative error for finite difference analysis of pressurized cylinder: standard (dashed) and modified (solid) formulation

Finite element model of segment of solid-fuel rocket using 4-node element with constant pressure

---

What is Locking?

First evidences: 1970’s – shear deformable beam/plate/shell at thin limit

- shear deformable plate and shell finite elements perform badly for high ratios \( t/a \)
- selective reduced integration of shear terms in the stiffness improves behavior
- influence of ‘parasitic’ shear is identified

Displacement at the center \( W \) vs. slenderness \( t/a \) for a simply supported square plate under uniform load \( q_0 \).

\[ \frac{WD}{a^4 q_0} \]

‘Parasitic’ shear stresses induced in a linear element under bending mode. (a) Constraint mode, (b) true mode.

\[ ^1 \text{Zienkiewicz, Taylor & Too (1971). Reduced integration technique in general analysis of plates and shells. International Journal for Numerical Methods in Engineering} \]
What is Locking?

**Definition**

*Locking means the effect of a reduced rate of convergence for coarse meshes in dependence of a critical parameter.*

**Symptoms of locking**

- solution is too stiff (displacements are too small)
- ‘parasitic’ strain mode
- oscillation of stresses/member forces
- dependency on a critical parameter

**Importance**

- good accuracy with coarse meshes
- improved behavior for large aspect ratio
- elimination of spurious stress/member force oscillations
- improved modeling capabilities for nearly incompressible materials (rubber), plastic flow, etc.
### What is Locking?

#### Examples of locking types

<table>
<thead>
<tr>
<th>Element types</th>
<th>Locking types</th>
<th>Critical parameter</th>
<th>Benchmark</th>
<th>Un-locking methods</th>
</tr>
</thead>
<tbody>
<tr>
<td>volumetric</td>
<td>in-plane shear</td>
<td>transverse shear</td>
<td>membrane</td>
<td>trapezoidal</td>
</tr>
<tr>
<td>axisymmetric, 2D plane strain and 3D solids</td>
<td>solid elements</td>
<td>shear deformable beams and shells</td>
<td>curved beam/shell</td>
<td>solid and solid-shell element</td>
</tr>
<tr>
<td>Poison ratio $\nu \rightarrow 0.5$</td>
<td>aspect ratio of element $t/l_e \rightarrow 0$</td>
<td>slenderness $t/a \rightarrow 0$</td>
<td>thickness-to-curvature $t/R \rightarrow 0$</td>
<td>thickness-to-curvature $t/R \rightarrow 0$</td>
</tr>
<tr>
<td>thick sphere under pressure, Cook membrane</td>
<td>bending of cantilever beam/plate</td>
<td>bending of cantilever beam/plate</td>
<td>Scordelis-Lo roof, pinched ring</td>
<td>Scordelis-Lo roof, pinched ring</td>
</tr>
<tr>
<td>IM, EAS, RI, B-bar</td>
<td>IM, EAS, RI</td>
<td>IM, EAS, RI, ANS</td>
<td>IM, EAS, RI, DSG</td>
<td>DSG</td>
</tr>
</tbody>
</table>

#### Material
- **IM** – incompatible mode method, covered in Lecture notes for L12?
- **EAS** – enhanced assumed strain method, covered in Lecture 13
- **RI** – reduced integration
- **B-bar** – B-bar
- **ANS** – assumed natural strain
- **DSG** – discrete strain gap method, out of scope of this course, see ref.¹

¹BLETZINGER, BISCHOFF & RAMM (2000). A unified approach for shear-locking-free triangular and rectangular shell finite elements. COMPUTERS & STRUCTURES
**Motivation**
- compatible
- variationally based locking-free element
- no zero energy modes
- less sensitive to distortions than IM

**Analysis of ‘parasitic’ strains for square finite element**

Displacement field

\[
\begin{align*}
    u &= xy \\
    v &= 0
\end{align*}
\]

Strain field

\[
\begin{align*}
    \varepsilon_{xx} &= u_x = y \\
    \varepsilon_{yy} &= v_y = 0 \\
    2\varepsilon_{xy} &= u_y + v_x = x
\end{align*}
\]

Material law

\[
C = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix}
1 - \nu & \nu & 0 \\
\nu & 1 - \nu & 0 \\
0 & 0 & \frac{1 - 2\nu}{2}
\end{bmatrix}
\]

Stress field

\[
\begin{align*}
    \sigma_{xx} &= \frac{E}{(1 + \nu)(1 - 2\nu)}((1 - \nu)\varepsilon_{xx} + \nu\varepsilon_{yy}) = \frac{E(1 - \nu)}{(1 + \nu)(1 - 2\nu)}y \\
    \sigma_{yy} &= \frac{E}{(1 + \nu)(1 - 2\nu)}((1 - \nu)\varepsilon_{yy} + \nu\varepsilon_{xx}) = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}y \\
    \sigma_{xy} &= \frac{E}{2(1 + \nu)}2\varepsilon_{xy} = \frac{E}{(1 + \nu)}x
\end{align*}
\]

...two wrongs do make a right in California' G. STRANG (1973)

...two rights make a right even in California' R. L. TAYLOR (1989)

\[
\begin{align*}
    \varepsilon_{xy} &\neq 0 \\
    \varepsilon_{xx} + \varepsilon_{yy} &> 0 \\
    \varepsilon_{xy} &\neq 0 \\
    \varepsilon_{xx} + \varepsilon_{yy} &\approx 0
\end{align*}
\]
Enhanced assumed strain (EAS) formulation in 2D (Simo & Rifai 1990)

### Deformation modes of a bilinear Q1, choice of EAS trial functions

<table>
<thead>
<tr>
<th>mode</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
<th>$c_4$</th>
<th>$c_5$</th>
<th>$c_6$</th>
<th>$c_7$</th>
<th>$c_8$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\alpha_3$</th>
<th>$\alpha_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u = $</td>
<td>1</td>
<td>0</td>
<td>$\xi$</td>
<td>0</td>
<td>$\eta$</td>
<td>0</td>
<td>$\xi\eta$</td>
<td>0</td>
<td>$\xi^2/2$</td>
<td>0</td>
<td>0</td>
<td>$\eta^2/2$</td>
</tr>
<tr>
<td>$v = $</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\xi$</td>
<td>0</td>
<td>$\eta$</td>
<td>0</td>
<td>$\xi\eta$</td>
<td>0</td>
<td>$\eta^2/2$</td>
<td>$\xi^2/2$</td>
<td>0</td>
</tr>
<tr>
<td>$\varepsilon_\xi$</td>
<td>0</td>
<td>0</td>
<td>$c_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$c_7\eta$</td>
<td>0</td>
<td>$\alpha_1\xi$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\varepsilon_\eta$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$c_6$</td>
<td>0</td>
<td>$c_8\xi$</td>
<td>0</td>
<td>$\alpha_2\eta$</td>
<td>0</td>
</tr>
<tr>
<td>$\varepsilon_{\xi\eta}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$c_4$</td>
<td>$c_5$</td>
<td>0</td>
<td>$c_7\xi$</td>
<td>$c_8\eta$</td>
<td>0</td>
<td>0</td>
<td>$\alpha_3\xi$</td>
<td>$\alpha_4\eta$</td>
</tr>
</tbody>
</table>

### Re-parametrized strain field

\[
\varepsilon^{\text{tot}} = \varepsilon^c + \ddot{\varepsilon} = \nabla^s N d_e + M\alpha = B d_e + M\alpha
\]

with standard strain-displacement matrix $B$ and trial function matrix $M$ of the enhanced strains $\alpha$

\[
M = \frac{J_0}{\dot{J}} T_0^{-1} \begin{bmatrix}
\xi & 0 & 0 & 0 \\
0 & \eta & 0 & 0 \\
0 & 0 & \xi & \eta \\
\end{bmatrix}
\]

\[
\alpha = \begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\end{bmatrix}
\]

transformation $M = \frac{J_0}{\dot{J}} T_0^{-1} \tilde{M}$ (explained later)
Enhanced assumed strain (EAS) formulation in 2D (Simo & Rifai 1990)

Variational basis is the Hu-Washizu principle with introduced re-parametrized strain field

$$
\Pi_{HW}(u, \tilde{\varepsilon}, \sigma) = \int_{\Omega} \left[ \frac{1}{2} (\nabla^s u + \tilde{\varepsilon})^T C (\nabla^s u + \tilde{\varepsilon}) - u^T b - \sigma \tilde{\varepsilon} \right] d V.
$$

Elimination of the stresses by orthogonality condition $\int_{\Omega} \sigma \tilde{\varepsilon} d V = 0$.

First variation leads to

$$
\delta \Pi_{HW} = \int_{\Omega} \left[ \delta (\nabla^s u + \tilde{\varepsilon})^T C (\nabla^s u + \tilde{\varepsilon}) - \delta u^T b \right] d V = 0.
$$

Introducing the discretization $\varepsilon^c = \nabla^s u = \nabla^s N d_e = B d_e$ and $\tilde{\varepsilon} = M \alpha$ results in

$$
\delta \Pi_{HW}^h = \delta d_e \left( K_e d_e + F_e^T \alpha_e - f_e \right) + \delta \alpha_e (F_e d_e + H_e \alpha_e) = 0
$$

with the abbreviations

$$
K_e = \int_{\Omega_e} B^T C B d V, \quad F_e = \int_{\Omega_e} M^T C B d V, \quad H_e = \int_{\Omega_e} M^T C M d V.
$$
With the fundamental lemma of variational calculus, we obtain the linear system of equations

\[
\begin{align*}
\delta \mathbf{d}_e : \quad & \mathbf{K}_e \mathbf{d}_e + \mathbf{F}_e^T \alpha_e = \mathbf{f}_e \\
\delta \alpha_e : \quad & \mathbf{F}_e \mathbf{d}_e + \mathbf{H}_e \alpha_e = 0 \\
\end{align*}
\Rightarrow \begin{bmatrix} \mathbf{K}_e & \mathbf{F}_e^T \\ \mathbf{F}_e & \mathbf{H}_e \end{bmatrix} \begin{bmatrix} \mathbf{d}_e \\ \alpha_e \end{bmatrix} = \begin{bmatrix} \mathbf{f}_e \\ \mathbf{0} \end{bmatrix}.
\]

By static condensation we obtain

\[
\mathbf{K}_e^{\text{EAS}} = \mathbf{K}_e - \mathbf{F}_e^T (\mathbf{H}_e)^{-1} \mathbf{F}_e.
\]

**Transformation of natural to global EAS strains**

additional strains are referred to the \(\xi, \eta\)-coordinate system (covariant transformation)

\[
\bar{\mathbf{\varepsilon}} = \begin{bmatrix} \tilde{\varepsilon}_{xx} & \tilde{\varepsilon}_{yy} & \tilde{\varepsilon}_{xy} \end{bmatrix}^T \rightarrow \tilde{\mathbf{\varepsilon}} = \begin{bmatrix} \tilde{\varepsilon}_{\xi\xi} & \tilde{\varepsilon}_{\eta\eta} & \tilde{\varepsilon}_{\xi\eta} \end{bmatrix}^T,
\]

\[
\tilde{\mathbf{\varepsilon}} = \frac{\mathbf{J}_0}{\mathbf{J}} \mathbf{T}_0^{-1} \bar{\mathbf{\varepsilon}}
\]

with

\[
\mathbf{T}_0 = \begin{bmatrix} x,\xi & x,\xi & y,\xi & x,\xi y,\xi \\ x,\eta & y,\eta & y,\eta & x,\eta y,\eta \\ 2x,\xi & 2y,\xi & x,\eta y,\xi + x,\xi x,\eta \end{bmatrix} = \begin{bmatrix} J_{11} J_{11} & J_{12} J_{12} & J_{11} J_{12} \\ J_{21} J_{21} & J_{22} J_{22} & J_{21} J_{22} \\ 2J_{11} J_{21} & 2J_{12} J_{22} & J_{21} J_{12} + J_{11} J_{22} \end{bmatrix}_{\xi=0}^{\eta=0}
\]

transformation of the trial function matrix \(\mathbf{M}\)

\[
\mathbf{M} = \frac{\mathbf{J}_0}{\mathbf{J}} \mathbf{T}_0^{-1} \bar{\mathbf{M}} \quad \text{where } \bar{\mathbf{M}} \text{ contains the trial functions in local coordinates}
\]
Enhanced assumed strain (EAS) formulation in 2D (Simo & Rifaï 1990)

**Enriched choice of more enhanced strain terms**

for improved behavior for distorted elements

\[ Q1E5 : \quad \tilde{\mathbf{M}} = \begin{bmatrix} \xi & 0 & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 \\ 0 & 0 & \xi & \eta & \xi \eta \end{bmatrix}, \quad Q1E7 : \quad \tilde{\mathbf{M}} = \begin{bmatrix} \xi & 0 & 0 & 0 & \xi \eta & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 & \xi \eta & 0 \\ 0 & 0 & \xi & \eta & 0 & 0 & \xi \eta \end{bmatrix}. \]

**Stress recovery**

recovery of the assumed strains and compatible strains

\[ \alpha = -\mathbf{H}^{-1}_e \mathbf{F}_e \mathbf{d}_e \quad \mathbf{\varepsilon}^c = \mathbf{B} \mathbf{d}_e \]

EAS strains

\[ \mathbf{\varepsilon}^{\text{EAS}} = \mathbf{B} \mathbf{d}_e + \mathbf{M} \alpha \]

stress recovery

\[ \mathbf{\sigma} = \mathbf{C} \mathbf{\varepsilon}^{\text{EAS}} \]

**Note about PLANE182 family**

1Simo & Rifaï (1990). A CLASS OF MIXED ASSUMED STRAIN METHODS AND THE METHOD OF INCOMPATIBLE MODES. IJNME
Enhanced assumed strain (EAS) formulation in 3D

Basic enhanced strain matrix

\[
\tilde{\mathbf{M}} = \begin{bmatrix}
0 & 0 & 0 & \xi & 0 & 0 & 0 & 0 & 0 \\
0 & \eta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \zeta & 0 & 0 & 0 & 0 \\
\xi & 0 & \eta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \xi & 0 & \zeta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \eta & \zeta & 0 \\
\end{bmatrix}
\]

Enriched choice of more enhanced strain terms

\[
\tilde{\mathbf{M}}_{10..21} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \xi \zeta & 0 & 0 & 0 & \xi \eta & 0 \\
0 & 0 & \eta \zeta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi \eta & 0 \\
0 & 0 & 0 & \eta \zeta & 0 & 0 & 0 & \xi \zeta & 0 & 0 & 0 & 0 \\
\xi \zeta & 0 & 0 & 0 & \eta \zeta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \xi \eta & 0 & 0 & 0 & 0 & \eta \zeta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \xi \eta & 0 & 0 & 0 & \xi \zeta & 0 & 0 & 0 \\
\end{bmatrix}
\]

Andelfinger & Ramm (1993). EAS-elements for two-dimensional, three-dimensional, plate and shell structures and their equivalence to HR-elements. IJNME
B-bar formulation

Starting point: standard stiffness matrix

\[ \mathbf{K}_e = \int_{\Omega_e} \mathbf{B}^T \mathbf{C} \mathbf{B} \, dV \]

with typical strain-displacement sub-matrix for node I

\[ \mathbf{B}_I = \begin{bmatrix} N_{I,x} & 0 \\ 0 & N_{I,y} \\ N_{I,y} & N_{I,x} \end{bmatrix} \]

Volumetric-deviatoric split

\[ \mathbf{\bar{B}}_I = \mathbf{B}_I^{\text{dev}} + \mathbf{B}_I^{\text{dil}} \]

with standard deviatoric part

\[ \mathbf{B}_I^{\text{dev}} = \begin{bmatrix} 0.5N_{I,x} & -0.5N_{I,y} \\ -0.5N_{I,x} & 0.5N_{I,y} \\ N_{I,y} & N_{I,x} \end{bmatrix} \]

and modified volumetric part for nearly-incompressible applications

\[ \mathbf{\bar{B}}_I^{\text{dil}} = \frac{1}{2} \begin{bmatrix} B_{Ix} & B_{Iy} \\ B_{Ix} & B_{Iy} \\ 0 & 0 \end{bmatrix} \]

with

\[ B_{Ix} = \frac{\int_{\Omega_e} N_{I,x} \, dV}{\int_{\Omega_e} dV}, \quad B_{Iy} = \frac{\int_{\Omega_e} N_{I,y} \, dV}{\int_{\Omega_e} dV} \]

**Note default formulation for PLANE182**

Hughes (1980). Generalization of selective integration procedures to anisotropic and nonlinear media. IJNME
B-bar formulation

Analysis of ‘parasitic’ strains for square finite element

volumetric-deviatoric split

$$\bar{\boldsymbol{B}}_I = \boldsymbol{B}^{\text{dev}}_I + \bar{\boldsymbol{B}}^{\text{dil}}_I$$

standard strain-displacement matrix

$$\boldsymbol{B} = \frac{1}{4} \begin{bmatrix} -1 + \eta & 0 & 1 - \eta & 0 & 1 + \eta & 0 & -1 - \eta & 0 \\ 0 & -1 + \xi & 0 & -1 - \xi & 0 & 1 + \xi & 0 & 1 - \xi \\ -1 + \xi & -1 + \eta & -1 - \xi & 1 - \eta & 1 + \xi & 1 + \eta & 1 - \xi & -1 - \eta \end{bmatrix}$$

standard deviatoric part

$$\boldsymbol{B}^{\text{dev}} = \frac{1}{8} \begin{bmatrix} -1 + \eta & 1 - \xi & 1 - \eta & 1 + \xi & 1 + \eta & -1 - \xi & -1 - \eta & -1 + \xi \\ 1 - \eta & -1 + \xi & -1 + \eta & -1 - \xi & -1 - \eta & 1 + \xi & 1 + \eta & 1 - \xi \\ -2 + 2\xi & -2 + 2\eta & -2 - 2\xi & 2 - 2\eta & 2 + 2\xi & 2 + 2\eta & 2 - 2\xi & -2 - 2\eta \end{bmatrix}$$

modified volumetric part

$$\bar{\boldsymbol{B}}^{\text{dil}} = \frac{1}{8} \begin{bmatrix} -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

application of in-plane bending deformation

xy-mode: $$\mathbf{d}_e = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}$$

$$\bar{\mathbf{B}}_I \mathbf{d}_e = \begin{bmatrix} y \\ -y \\ x \end{bmatrix}$$

$$\boldsymbol{\sigma} = \mathbf{C} \bar{\mathbf{B}}_I \mathbf{d}_e = \frac{E}{2(1+\nu)} \begin{bmatrix} y \\ -y \\ x \end{bmatrix}$$
Reduced integration and hourglass stabilization

Reduced integration

\[ K_e = \int_{\Omega_e} B^T C B \, dV \approx B^T C B \, \det J_{|\xi=0, \eta=0} \cdot 4 = K_e^{RI} \]

Rang of the numerically integrated stiffness matrix

\[ \text{rank } K_e^{RI} = 3 \quad \text{whereas correct rank is } \quad n_{\text{dof}} - n_{\text{rbm}} = 5 \]

Local eigenvalues for square element: reduced integration and full integration

\[
\begin{array}{ccc}
0 & 0 & 0 \\
30 & 30 & 30 \\
0 & 15 & 0
\end{array}
\]

\[
\begin{array}{ccc}
0 & 0 & 0 \\
30 & 30 & 30 \\
0 & 15 & 0
\end{array}
\]

\[
\begin{array}{ccc}
E = 30 \\
\nu = 0.0 \\
\text{thickness} = 1
\end{array}
\]

plain stress

\[ \text{spurious zero energy modes (ZEM)} \]
Reduced integration and hourglass stabilization

dimensions:
• length = 25 mm
• breadth = 5 mm
• height = 5 mm

material:
• linear elastic steel

boundary conditions:
• A = fixed support
• B = prescribed boundary motion
  (6 mm vertical displacement)

uniform reduced integration 1 GP

uniform reduced integration 1GP with hourglass stabilization
Reduced integration and hourglass stabilization

The stiffness matrix consists of a one-point quadrature matrix $\mathbf{K}_{e}^{RI}$ and stabilization matrix $\mathbf{K}_{e}^{stab}$

$$\mathbf{K}_{e} = \mathbf{K}_{e}^{RI} + \mathbf{K}_{e}^{stab}$$

The stabilization matrix is constructed by

$$\mathbf{K}_{e}^{stab} = \int_{\Omega_{e}} \mathbf{\tilde{B}}^{T} \mathbf{C} \mathbf{\tilde{B}} \, dV$$

with the strain-displacement matrix for node $I$ \[
\mathbf{\tilde{B}}_{I} = \begin{bmatrix}
    e_{1} h_{,x} \gamma^{T} & e_{2} h_{,y} \gamma^{T} \\
    e_{2} h_{,x} \gamma^{T} & e_{1} h_{,y} \gamma^{T} \\
    e_{3} h_{,y} \gamma^{T} & e_{3} h_{,x} \gamma^{T}
\end{bmatrix}
\]

with $e_{1}$, $e_{2}$, $e_{3}$ as free parameters

with $h = \xi \eta$

with $\gamma = \frac{1}{4} \left[ h - (\mathbf{h}^{T} \mathbf{x}) \mathbf{b}_{x} - (\mathbf{h}^{T} \mathbf{y}) \mathbf{b}_{y} \right]$ with hourglass mode $\mathbf{h}^{T} = \begin{bmatrix} +1 & -1 & +1 & -1 \end{bmatrix}$

with strain-displacement vectors \[
\mathbf{b}_{x}^{T} = \mathbf{N}_{x} = \begin{bmatrix} N_{1,x} & N_{2,x} & N_{3,x} & N_{4,x} \end{bmatrix} \\
\mathbf{b}_{y}^{T} = \mathbf{N}_{y} = \begin{bmatrix} N_{1,y} & N_{2,y} & N_{3,y} & N_{4,y} \end{bmatrix}
\]

$x$ and $y$ are nodal coordinate vectors
Comparison cost of different formulations of Q1 family

<table>
<thead>
<tr>
<th>Element type</th>
<th>rank $K_e$</th>
<th>locking</th>
<th>cost</th>
<th>relative to Q1 2x2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q1 2x2 (full) integration</td>
<td>5 (ok)</td>
<td>yes</td>
<td>yes</td>
<td>3040</td>
</tr>
<tr>
<td>Q1RI 1x1</td>
<td>3 (2 ZEM)</td>
<td>no</td>
<td>no</td>
<td>476</td>
</tr>
<tr>
<td>Q1RI 1x1 + stab*</td>
<td>5 (ok)</td>
<td>no</td>
<td>no</td>
<td>3536</td>
</tr>
<tr>
<td>Q1E4</td>
<td>5 (ok)</td>
<td>no</td>
<td>no</td>
<td>5156</td>
</tr>
<tr>
<td>Q1E5</td>
<td>5 (ok)</td>
<td>no</td>
<td>no</td>
<td>5656</td>
</tr>
<tr>
<td>Q1E7</td>
<td>5 (ok)</td>
<td>no</td>
<td>no</td>
<td>7907</td>
</tr>
</tbody>
</table>

**Assumptions**

- no optimization of operation and no use of matrix symmetries is assumed.
- matrix summation $A$ and $B$ ($nxm$) costs $n*m$ [flops]
- matrix multiplication $A$ ($nxm$) and $B$ ($mxk$) is $n*m*k$ [flops] (naïve algorithm)
- operation count for chain matrix multiplication
  \[ A.B.C = (A.B).C = (nxm).(mxk).(kxl) = n*m*k + n*l*k = (l+m)*n*k \]
- operation count for matrix inversion $A$ ($nxn$) is assumed $4*n^3$ [ops]
- calculation of Jacobian - 4 items each cost 16 operation = 64 [ops]
- cost of material procedure (if necessary) is neglected
- multiplications with transformation matrix T0 at EAS is included

*USED IN NON-LINEAR EXPLICIT TIME INTEGRATION WHERE THE STIFFNESS MATRIX NOT USED*
Practical importance of locking-free finite elements

Eigenvalue analysis of an annular plate \((K - \omega^2M)\phi = 0\)

\[
\begin{align*}
  r_i &= 0.8 \\
  r_a &= 1.0 \\
\end{align*}
\]
lin. elastic material, plane strain
\(E = 1000\)
\(\nu = 0.49\)
\(\rho = 1\)

- reference solution: 180x30 Q2 elements, mixed u/p formulation
- comparison of 18x3 element mesh of
  - PLANE 42, full integration with locking!
  - PLANE 182, full integration + B-bar formulation
  - PLANE 182, uniform reduced int. + hourglass stab.
  - PLANE 182, EAS formulation

<table>
<thead>
<tr>
<th>REFERENCE</th>
<th>PLANE 42, full int.</th>
<th>PLANE 182, Bbar</th>
<th>PLANE 182, RI</th>
<th>PLANE 182, EAS</th>
</tr>
</thead>
<tbody>
<tr>
<td>mode no.</td>
<td>frequency</td>
<td>frequency</td>
<td>err. in %</td>
<td>frequency</td>
</tr>
<tr>
<td>1</td>
<td>1.6442</td>
<td>2.5929</td>
<td>57.6998</td>
<td>1.7763</td>
</tr>
<tr>
<td>3</td>
<td>5.6580</td>
<td>7.5946</td>
<td>34.2276</td>
<td>6.1776</td>
</tr>
</tbody>
</table>
Practical importance of locking-free finite elements

When does volumetric locking come into play?

Plastic flow for J2 plasticity (von Mises) is isochoric

\[ f = |\text{dev } \sigma| - \sigma_y \]
\[ \dot{\varepsilon}^P = \lambda \frac{\text{dev } \sigma}{|\text{dev } \sigma|} \]

Elasto-plastic problem for a plate with a hole*

![Diagram of a plate with a hole](image)

- E = 70 GPa
- \( \nu = 0.3 \)
- \( \sigma_y = 243 \text{ MPa} \)
- H = 20 MPa
- plane strain!

*Problem data taken from p.96 „Finite Element Method for Solid and Structural Mechanics“ Zienkiewicz and Taylor
Practical importance of locking-free finite elements

Model definition with ANSYS

- Pilot node prescribes displacement on the upper edge
- Maximum value $u_y^{\text{max}} = 3.6 \text{ mm}$

Symmetry
Practical importance of locking-free finite elements

equivalent plastic strains for different formulations!
different picture for CST! Q1 does not completely catch it!
Important elements that are out of scope of this lecture

- methods against membrane, trapezoidal
- U-p formulation\(^1\)
  - H (Abaqus), SOLID18X family (ANSYS)
- averaged nodal pressure tetrahedron\(^2\)
  - C3D4H (Abaqus), SOLID285 (ANSYS), elform=13 (LS-Dyna)
- nodally integrated tetrahedrons\(^3\)
- linked interpolation against locking (2-node Timoshenko beam\(^4\), plate elements\(^5,6\))
- isogeometric (NURBS-based) beams\(^7,8\), shells\(^9,10\) and solids\(^11\)

REFERENCES ARE NOT COMPLETE, COLLECTED TO MY TASTE

\(^7\)Echter, R., & Bischoff, M. (2010). Numerical efficiency, locking and unlocking of NURBS finite elements. CMAME
\(^8\)Bouclier, R., Elgueudj, T., & Combescure, A. (2012). Locking free isogeometric formulations of curved thick beams. CMAME
Lecture 13: Finite element method VII

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