CTU, Prague
05.06.2018

Lecture 8: Finite Element Method III
Mass matrices

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Properties of mass matrix

Consistent and lumped mass matrix

High-order mass matrix

Direct inversion of mass matrix

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Baustatik und Baudynamik

Motivation for mass matrices

- discrete representation of inertia properties of structures
- needed for transient analysis $\mathbf{M}\ddot{\mathbf{U}} = \mathbf{F}^{\text{ext}} - \mathbf{F}^{\text{int}}(\mathbf{U})$ Lectures 17 and 19
- modal and spectral analyses $(\mathbf{K} - \omega^2\mathbf{M})\phi = \mathbf{0}$ Lecture 16
- harmonic analysis $(\mathbf{K} + i\omega\mathbf{C} - \omega^2\mathbf{M})\mathbf{U} = \mathbf{F}$

- gravity load and Rayleigh model of damping Lecture 22

$$\mathbf{F}^{\text{grav}} = \mathbf{M}\mathbf{g} \qquad \mathbf{C} = a_k\mathbf{K} + a_m\mathbf{M}$$

Mass matrices influence

- computational cost in explicit transient and modal analyses Lectures 17 and 19
- accuracy through discretization (dispersion) error Lectures 8 and 16

Understanding and choosing mass matrices is important!

Properties of mass matrix

Physical properties

- matrix symmetry
- positive definiteness (semi-positive)
- preservation of total translational mass
- physical symmetry

$$\mathbf{M} = \mathbf{M}^{\mathrm{T}}$$

$$\mathbf{v}^{\mathrm{T}}\mathbf{M}\mathbf{v} \geq 0 \quad \mathbf{v} \neq \mathbf{0}$$

$$\int_{\Omega} \rho \, \mathrm{d}\, V = \mathbf{1}_{\mathbf{x}}^{\mathrm{T}} \, \mathbf{M} \, \mathbf{1}_{\mathbf{x}}$$

with unit motion in x-,y- or z-direction

$$\mathbf{1}_{\mathbf{x}} = [1,0,0,1,0,0,1,0,0,\cdots]$$

Algorithmic properties

- low fill-in or sparsity (ideally diagonal)
- easiness of computation

Consistent mass matrix for continua

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Virtual work principle

$$\int_{\Omega} \left(\delta \mathbf{u}^{\mathrm{T}} \rho \ddot{\mathbf{u}} + \delta \boldsymbol{\varepsilon}^{\mathrm{T}} \boldsymbol{\sigma} - \delta \mathbf{u}^{\mathrm{T}} \hat{\mathbf{b}} \right) \mathrm{d}V - \int_{\Gamma_\sigma} \delta \mathbf{u}^{\mathrm{T}} \hat{\mathbf{t}} \, \mathrm{d}A = 0$$

Spatial discretization

$$\mathbf{u} \approx \mathbf{u}^{\mathrm{h}} = \mathbf{N}\mathbf{U} \qquad \delta\mathbf{u} \approx \delta\mathbf{u}^{\mathrm{h}} = \mathbf{N}\delta\mathbf{U} \qquad \boldsymbol{\varepsilon} \approx \boldsymbol{\varepsilon}^{\mathrm{h}} = \mathbf{B}\mathbf{U} \qquad \delta\boldsymbol{\varepsilon} \approx \delta\boldsymbol{\varepsilon}^{\mathrm{h}} = \mathbf{B}\delta\mathbf{U}$$

Dynamic equation of motion

$$\mathbf{M}\ddot{\mathbf{U}} = \mathbf{F}^{\text{ext}} - \mathbf{F}^{\text{int}}(\mathbf{U})$$

Consistent mass matrix (CMM)

$$\mathbf{M} = \int_{\Omega} \rho \mathbf{N}^{\mathrm{T}} \mathbf{N} \, \mathrm{d}V$$

Internal and external force vectors

$$\mathbf{F}^{\text{int}} = \int_{\Omega} \mathbf{B}^{\mathrm{T}} \boldsymbol{\sigma} \, \mathrm{d}V$$

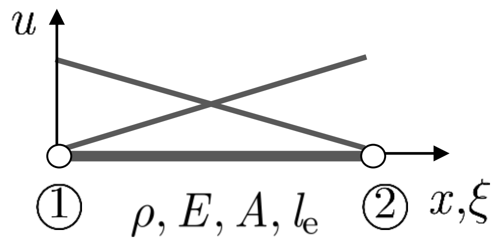
$$\mathbf{F}^{\text{ext}} = \int_{\Omega} \mathbf{N}^{\mathrm{T}} \hat{\mathbf{b}} \, \mathrm{d}V \int_{\Gamma_\sigma} \mathbf{N}^{\mathrm{T}} \hat{\mathbf{t}} \, \mathrm{d}A$$

Example: 2-node truss in 1D

Shape function in isoparametric coordinates

$$N_1 = \frac{1}{2}(1-\xi)$$

$$N_2 = \frac{1}{2}(1+\xi)$$



Consistent mass matrix on element level
(constant density and cross-section)

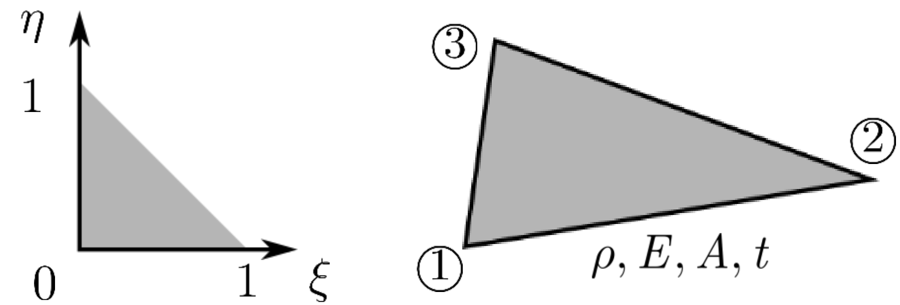
$$\mathbf{m}_\mathrm{e} = \int_0^{l_\mathrm{e}} \rho \mathbf{N}^\mathrm{T}\mathbf{N}\,\mathrm{d}x$$

$$= \int_{-1}^{1} \frac{\rho l_\mathrm{e}}{2} \begin{bmatrix} N_1 \cdot N_1 & N_1 \cdot N_2 \\ N_2 \cdot N_1 & N_2 \cdot N_2 \end{bmatrix} \mathrm{d}\xi = \frac{\rho A l_\mathrm{e}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Example: 3-node triangle in 2D

Shape function in isoparametric coordinates

$$N_1 = 1 - \xi - \eta \quad N_2 = \xi \quad N_3 = \eta$$



Consistent mass matrix on element level
(constant density)

$$\mathbf{m}_\mathrm{e} = \int_A \rho \mathbf{N}^\mathrm{T}\mathbf{N}\,t\,\mathrm{d}A$$

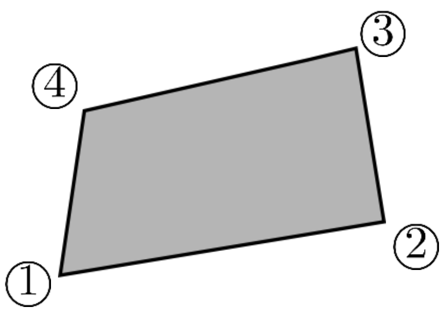
$$= \frac{\rho A t}{12} \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix}$$

Consistent mass matrix for continua

Example: 4-node quadrilateral in 2D

Shape function in isoparametric coordinates

$$N_i = \frac{1}{2}(1+\xi_i\xi)(1+\eta_i\eta)$$



Consistent mass matrix on element level
via 2x2 numerical quadrature (Gauss)

$$\mathbf{m}_\mathrm{e} = \int\limits_{\Omega_\mathrm{e}} \rho \mathbf{N}^\mathrm{T}\mathbf{N}\, t \,\mathrm{d}x\,\mathrm{d}y = \int\limits_{-1}^{1}\int\limits_{-1}^{1} \rho \mathbf{N}^\mathrm{T}\mathbf{N} |J|\, t\,\mathrm{d}\xi\mathrm{d}\eta$$

$$\approx \sum_{GP=1}^{4} \left[(\rho \mathbf{N}^\mathrm{T}\mathbf{N}|J|t)|_{\substack{\xi=\xi_{GP}\\ \eta=\eta_{GP}}} w_{GP} \right]$$

Note: 8-node hexahedron requires
3x3x3 quadrature

Properties of consistent mass matrix

- matrix symmetry (by construction)

$$\mathbf{M} = \mathbf{M}^\mathrm{T}$$

- positive definiteness (for full integration)

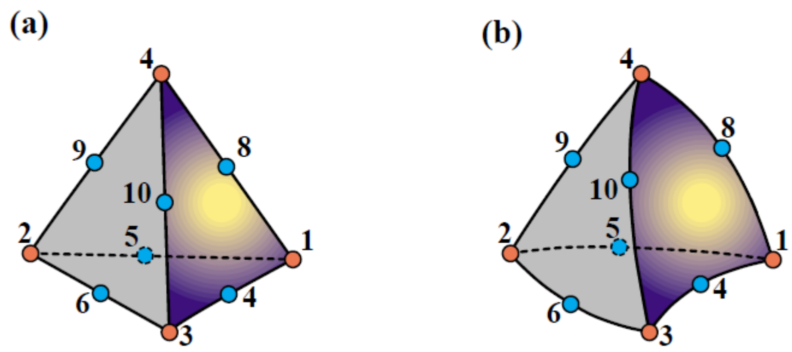
$$\mathbf{v}^\mathrm{T}\mathbf{M}\mathbf{v} \geq 0 \quad \mathbf{v} \neq \mathbf{0}$$

- preservation of total translational mass
  (for full integration)

$$\int\limits_{\Omega} \rho \,\mathrm{d}V = \mathbf{1}_\mathbf{x}^\mathrm{T}\,\mathbf{M}\,\mathbf{1}_\mathbf{x}$$

- physical symmetry
- low fill-in but non-diagonal
- easiness of computation

Example: 10-node tetrahedral in 3D




```
w(1:14,1) = [ ...
  0.13283874668559071814, ...
  0.13283874668559071814, ...
  0.13283874668559071814, ...
  0.13283874668559071814, ...
  0.088589824742980710434, ...
  0.088589824742980710434, ...
  0.088589824742980710434, ...
  0.088589824742980710434, ...
  0.019047619047619047619, ...
  0.019047619047619047619, ...
  0.019047619047619047619, ...
  0.019047619047619047619, ...
  0.019047619047619047619, ...
  0.019047619047619047619 ];

xyz(1:3,1:14) = [ ...
  0.056881379520423421748,  0.31437287349319219275,  0.31437287349319219275; ...
  0.31437287349319219275,  0.056881379520423421748,  0.31437287349319219275; ...
  0.31437287349319219275,  0.31437287349319219275,  0.056881379520423421748; ...
  0.31437287349319219275,  0.31437287349319219275,  0.31437287349319219275; ...
  0.69841970432438656092,  0.10052676522520447969,  0.10052676522520447969; ...
  0.10052676522520447969,  0.69841970432438656092,  0.10052676522520447969; ...
  0.10052676522520447969,  0.10052676522520447969,  0.69841970432438656092; ...
  0.10052676522520447969,  0.10052676522520447969,  0.10052676522520447969; ...
  0.50000000000000000000,  0.50000000000000000000,  0.00000000000000000000; ...
  0.50000000000000000000,  0.00000000000000000000,  0.50000000000000000000; ...
  0.50000000000000000000,  0.00000000000000000000,  0.00000000000000000000; ...
  0.00000000000000000000,  0.50000000000000000000,  0.50000000000000000000; ...
  0.00000000000000000000,  0.50000000000000000000,  0.00000000000000000000; ...
  0.00000000000000000000,  0.00000000000000000000,  0.50000000000000000000 ]';
```

Tensor products rule are not symmetric and leading to physically unsymmetric matrix. No closed expression of the quadrature are available → numerical search, e.g. Felippa rules

CARLOS FELIPPA, A COMPENDIUM OF FEM INTEGRATION FORMULAS FOR SYMBOLIC WORK, ENGINEERING COMPUTATION, VOLUME 21, NUMBER 8, 2004, PAGES 867-890.

AVAILABLE AS FORTRAN AND MATLAB CODE FOR 1-, 4-, 8-, 14-, 15-, 24-POINT R FROM HTTPS://PEOPLE.SC.FSU.EDU/~JBURKARDT/M_SRC/TETRAHEDRON_FELIPPA_RULE/TETRAHEDRON_FELIPPA_RULE.HTML

Lumped mass matrix

Motivation for lumped (diagonalized) mass matrices

- trivial computation of acceleration from the force vector $\ddot{\mathbf{U}} = \mathbf{M}^{-1}(\mathbf{F}^{\text{ext}} - \mathbf{F}^{\text{int}}(\mathbf{U}))$
- increased critical time step with respect to consistent mass matrix Lecture 17
- reduced storage in RAM
- reduced cost in modal analysis $(\mathbf{K} - \omega^2\mathbf{M})\phi = \mathbf{0}$

Methods for mass lumping (diagonalization)

- row-sum-lumping
- Hinton-Rock-Zienkiewicz method (1976)
- nodal quadrature for elements with Gauss-Lobatto nodal location

Lumped mass matrix

Row-sum-lumping

1. Starting point: consistent mass matrix
2. Add all terms at each row to diagonal

$$s_i = \sum_{j=1}^{n} m_{\mathrm{e},ij}$$

$$\mathbf{m}_\mathrm{e}^\mathrm{D} = \mathrm{diag}(s_i)$$

Example: 2-node truss in 1D

Consistent mass matrix on element level (constant density and cross-section)

$$\mathbf{m}_\mathrm{e} = \frac{\rho A l_\mathrm{e}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Lumped mass matrix by row-sum-diagonalization

$$\mathbf{m}_\mathrm{e}^\mathrm{D} = \frac{\rho A l_\mathrm{e}}{6} \begin{bmatrix} 2+1 & 0 \\ 0 & 1+2 \end{bmatrix} = \frac{\rho A l_\mathrm{e}}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example: 3-node triangle in 2D

Consistent mass matrix on element level (constant density)

$$\mathbf{m}_\mathrm{e} = \frac{\rho A t}{12} \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix}$$

Lumped mass matrix by row-sum-diagonalization

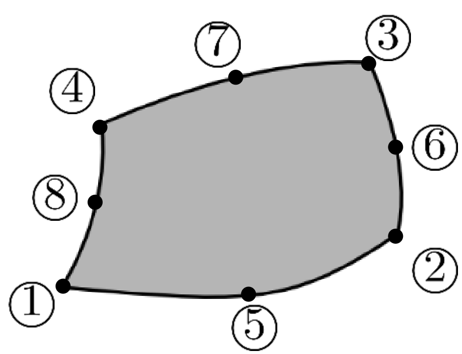
$$\mathbf{m}_\mathrm{e}^\mathrm{D} = \frac{\rho A t}{3} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Lumped mass matrix

Example: 8-node quadrilateral in 2D (Serendipity)

Shape function in isoparametric coordinates

$$N_1 = -0.25(1-\xi)(1-\eta)(1+\xi+\eta)$$
$$N_2 = -0.25(1+\xi)(1-\eta)(1-\xi+\eta)$$
$$N_3 = -0.25(1+\xi)(1+\eta)(1-\xi-\eta)$$
$$N_4 = -0.25(1-\xi)(1+\eta)(1+\xi-\eta)$$
$$N_5 = 0.5(1-\xi^2)(1-\eta)$$
$$N_6 = 0.5(1-\eta^2)(1+\xi)$$
$$N_7 = 0.5(1-\xi^2)(1+\eta)$$
$$N_8 = 0.5(1-\eta^2)(1-\xi)$$

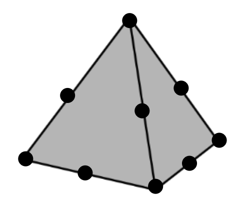


Lumped mass matrix by row-sum-diagonalization (constant density and Jacobian)

$$\mathbf{m}_{\mathrm{e}}^{\mathrm{D}} = \frac{\rho A t}{12}\,\mathrm{diag}(\underbrace{-1,-1,-1,-1}_{},4,4,4,4)$$

negative terms at corner nodes!

The same problem is observed for quadratic 10-node tetrahedral finite element



Lumped mass matrix by row-sum-diagonalization (constant density and Jacobian)

$$\mathbf{m}_{\mathrm{e}}^{\mathrm{D}} = \frac{\rho V}{20}\,\mathrm{diag}(\underbrace{-1,-1,-1,-1}_{},4,4,4,4,4,4)$$

negative terms at corner nodes!

Remedy: Hinton-Rock-Zienkiewicz method

Lumped mass matrix

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Hinton-Rock-Zienkiewicz method

1. Starting point: consistent mass matrix
2. Compute some of all diagonal terms of CMM for one spatial direction

$$S = \sum_{i=1}^{n} m_{\mathrm{e},ii}$$

3. Compute diagonal terms as total mass of the elements scaled with ratio of diagonal term to the sum of diagonal terms

$$s_i = \frac{m_{\mathrm{e},ii}}{S} m_e$$

$$\mathbf{m}_{\mathrm{e}}^{\mathrm{D}} = \mathrm{diag}(s_i)$$

Example: 8-node quadrilateral in 2D (Serendipity)

Lumped mass matrix by HRZ lumping (constant density and Jacobian)

$$\mathbf{m}_{\mathrm{e}}^{\mathrm{D}} = \frac{\rho A t}{36} \mathrm{diag}(1,1,1,1,8,8,8,8)$$

Example: quadratic 10-node tetrahedral finite element

Lumped mass matrix by HRZ lumping (constant density and Jacobian)

$$\mathbf{m}_{\mathrm{e}}^{\mathrm{D}} = \frac{\rho V}{108} \mathrm{diag}(3,3,3,3,16,16,16,16,16,16)$$

Motivation for high-order mass matrices

- the eigenfrequencies for CMM tend to be higher than analytical values
- the eigenfrequencies for LMM tend to be lower than analytical values
- weighted some of both can cancel the error and yield better results

Example: 2-node truss in 1D

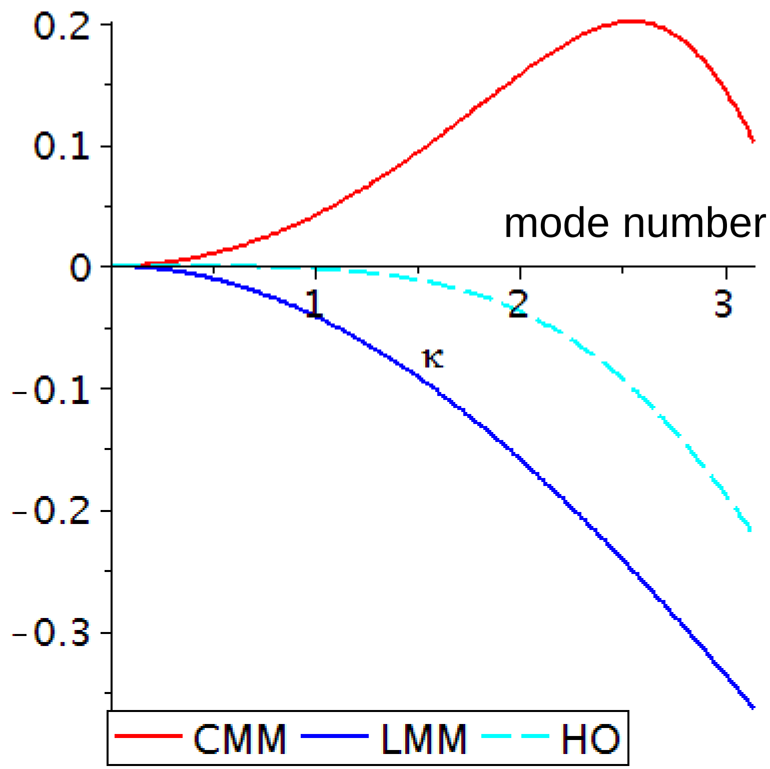
Consistent and lumped mass matrices
(constant density and cross-section)

$$\mathbf{m}_{\mathrm{e}} = \frac{\rho A l_{\mathrm{e}}}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{m}_{\mathrm{e}}^{\mathrm{D}} = \frac{\rho A l_{\mathrm{e}}}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Lumped mass matrix by row-sum-diagonalization

$$\mathbf{m}_{\mathrm{e}}^{\mathrm{HO}} = 0.5\mathbf{m}_{\mathrm{e}} + 0.5\mathbf{m}_{\mathrm{e}}^{\mathrm{D}} = \frac{\rho A l_{\mathrm{e}}}{12}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$$

frequency relative error



Example: 2-node truss in 1D

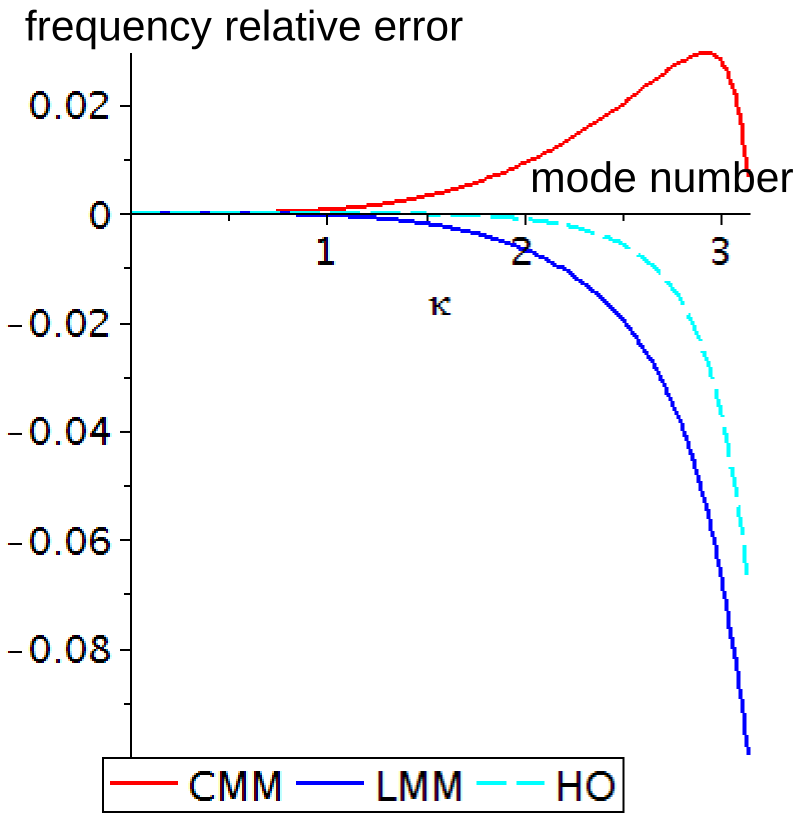
Shape function in isoparametric coordinates

$$N_1 = -0.5(1-\xi)\xi$$
$$N_2 = 0.5(1+\xi)\xi$$
$$N_3 = 1-\xi^2$$

Consistent and lumped mass matrices (constant density and cross-section)

$$\mathbf{m}_\mathrm{e} = \frac{\rho A l_\mathrm{e}}{30}\begin{bmatrix} 4 & -1 & 2 \\ -1 & 4 & 2 \\ 2 & 2 & 16 \end{bmatrix}$$

$$\mathbf{m}_\mathrm{e}^\mathrm{D} = \frac{\rho A l_\mathrm{e}}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$



Lumped mass matrix by row-sum-diagonalization

$$\mathbf{m}_\mathrm{e}^\mathrm{HO} = \frac{1}{3}\mathbf{m}_\mathrm{e} + \frac{2}{3}\mathbf{m}_\mathrm{e}^\mathrm{D} = \frac{\rho A l_\mathrm{e}}{90}\begin{bmatrix} 14 & -1 & 2 \\ -1 & 14 & 2 \\ 2 & 2 & 56 \end{bmatrix}$$

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Motivation for reciprocal mass matrices

- trivial computation of acceleration from the force vector
- increased critical time step with respect to lumped mass matrix
- memory storage comparable with CMM
- more accurate than HRZ lumping for elements without stable row-sum-diagonalizetion

$$\ddot{\mathbf{U}} = \mathbf{C}^{\circ}(\mathbf{F}^{\mathrm{ext}} - \mathbf{F}^{\mathrm{int}}(\mathbf{U}))$$

"directly" constructed inverse mass

Virtual work principle

$$\int_{\Omega} \left(\delta\mathbf{u}^{\mathrm{T}}\rho\ddot{\mathbf{u}} + \delta\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\sigma} - \delta\mathbf{u}^{\mathrm{T}}\hat{\mathbf{b}}\right)\,\mathrm{d}V - \int_{\Gamma_\sigma} \delta\mathbf{u}^{\mathrm{T}}\hat{\mathbf{t}}\,\mathrm{d}A = 0$$

"recast the term"

$$\delta\mathbf{u}^{\mathrm{T}}\dot{\mathbf{p}}$$

$\rho\dot{\mathbf{u}} = \mathbf{p}$ (linear momentum or impulse)

$$\delta\mathbf{u}^{\mathrm{T}}\dot{\mathbf{p}} + \delta\mathbf{p}^{\mathrm{T}}\left(\dot{\mathbf{u}} - \rho^{-1}\mathbf{p}\right)$$

$$\delta\mathbf{u}^{\mathrm{T}}\dot{\mathbf{p}} + \delta\mathbf{p}^{\mathrm{T}}\left(\dot{\mathbf{u}} - (1 - C_2)\rho^{-1}\mathbf{p} - C_2\mathbf{v}\right) + \delta\mathbf{v}^{\mathrm{T}}C_2(\rho\dot{\mathbf{v}} - \mathbf{p})$$

$\dot{\mathbf{u}} = \mathbf{v}$ (velocity)

C_2 (free parameter)

TKACHUK, A., BISCHOFF, M. (2015).

Parametrized virtual work principle

$$\int\limits_{\Omega} \left(\delta\mathbf{u}^{\mathrm{T}}\dot{\mathbf{p}} + \delta\mathbf{p}^{\mathrm{T}}\left(\dot{\mathbf{u}} - (1-C_2)\rho^{-1}\mathbf{p} - C_2\mathbf{v}\right) + \delta\mathbf{v}^{\mathrm{T}}C_2(\rho\dot{\mathbf{v}} - \mathbf{p})\right)\,\mathrm{d}V$$
$$+\int\limits_{\Omega}\left(\delta\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\sigma} - \delta\mathbf{u}^{\mathrm{T}}\hat{\mathbf{b}}\right)\,\mathrm{d}V - \int\limits_{\Gamma_\sigma}\delta\mathbf{u}^{\mathrm{T}}\hat{\mathbf{t}}\,\mathrm{d}A = 0$$

Spatial discretization

$$\mathbf{v} \approx \mathbf{v}^{\mathrm{h}} = \boldsymbol{\Psi}\mathbf{V} \qquad\qquad \mathbf{p} \approx \mathbf{p}^{\mathrm{h}} = \boldsymbol{\chi}\mathbf{P}$$

Dynamic equation of motion

$$\begin{cases} \mathbf{A}\dot{\mathbf{P}} &= \mathbf{F}^{\mathrm{ext}} - \mathbf{F}^{\mathrm{int}} \\ \mathbf{A}^{\mathrm{T}}\dot{\mathbf{U}} &= (1-C_2)\mathbf{C}\mathbf{P} + C_2\mathbf{W}\mathbf{V} \\ \mathbf{Y}\mathbf{V} &= \mathbf{W}^{\mathrm{T}}\mathbf{P} \end{cases}$$

Spatial discretization

$\mathbf{A} = \int\limits_{\Omega}\mathbf{N}^{\mathrm{T}}\boldsymbol{\chi}^{\mathrm{T}}\,\mathrm{d}\Omega,$ projection

$\mathbf{C} = \int\limits_{\Omega}\rho^{-1}\boldsymbol{\chi}^{\mathrm{T}}\boldsymbol{\chi}\,\mathrm{d}\Omega,$ reciprocal mass matrix

$\mathbf{Y} = \int\limits_{\Omega}\rho\boldsymbol{\Psi}^{\mathrm{T}}\boldsymbol{\Psi}\,\mathrm{d}\Omega,$ mass with $\mathbf{v}$

$\mathbf{W} = \int\limits_{\Omega}\boldsymbol{\chi}^{\mathrm{T}}\boldsymbol{\Psi}\,\mathrm{d}\Omega$ projection

Reciprocal mass matrices

$$\begin{cases} \mathbf{A}\dot{\mathbf{P}} &= \mathbf{F}^{\text{ext}} - \mathbf{F}^{\text{int}} \\ \mathbf{A}^{\mathrm{T}}\dot{\mathbf{U}} &= (1 - C_2)\mathbf{C}\mathbf{P} + C_2\mathbf{W}\mathbf{V} \\ \mathbf{Y}\mathbf{V} &= \mathbf{W}^{\mathrm{T}}\mathbf{P} \end{cases}$$

Elimination of velocity and momentum degrees of freedom by static condensation

$$\begin{cases} \mathbf{V} = \mathbf{Y}^{-1}\mathbf{W}^{\mathrm{T}}\mathbf{P} \\ \dot{\mathbf{U}} = \mathbf{C}^{\circ}\mathbf{P} \end{cases}$$

biorthogonality condition

$$A_{ij} = \int_{\Omega} N_i \chi_j \, \mathrm{d}\Omega = \delta_{ij} \qquad \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\ddot{\mathbf{U}} = \mathbf{C}^{\circ}(\mathbf{F}^{\text{ext}} - \mathbf{F}^{\text{int}})$$

$$\mathbf{C}^{\circ} = \mathbf{C} + C_2(\mathbf{W}\mathbf{Y}^{-1}\mathbf{W}^{\mathrm{T}} - \mathbf{C})$$

Dynamic equation of motion
with scaled reciprocal mass matrix

Positive definite:

$$\mathbf{C}^{\circ} = \mathbf{C} + \tilde{\boldsymbol{\lambda}}^{\circ} = (1 - C_2)\underbrace{\mathbf{C}} + C_2\underbrace{\mathbf{W}\mathbf{Y}^{-1}\mathbf{W}^{\mathrm{T}}} \qquad 0 < C_2 < 1$$

positive definite

Sparse fill-in of CMM:

$\mathbf{Y}$ is block diagonal for element-wise interpolation of $\mathbf{v}$

TKACHUK, A., BISCHOFF, M. (2015).

velocity shape functions $\mathbf{\Psi}$

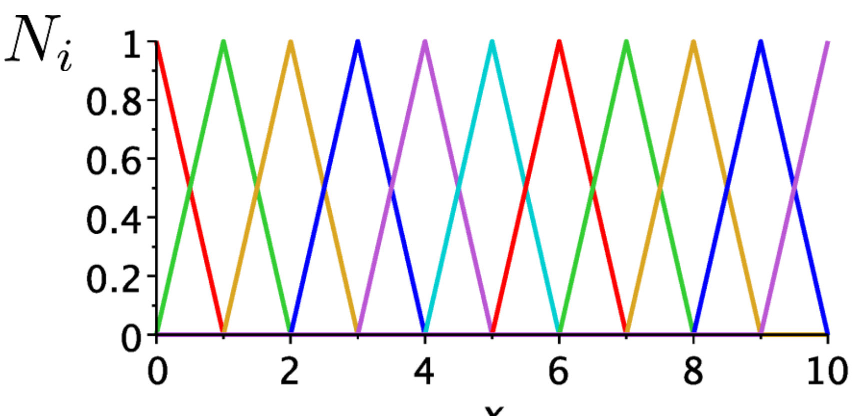
1D: $\mathbf{\Psi} = \begin{bmatrix} 1 \end{bmatrix}$ 2D: $\mathbf{\Psi} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ element-wise interpolation

linear momentum shape functions $\boldsymbol{\chi}$

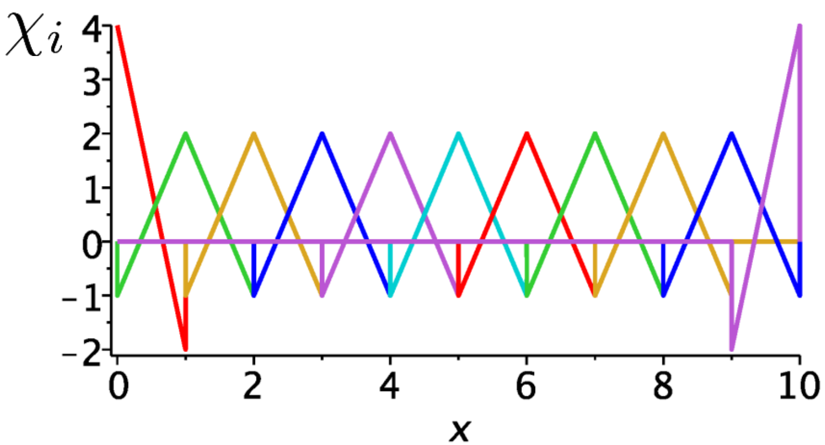
biorthogonality condition

$$A_{ij} = \int_{\Omega} N_i \chi_j \, \mathrm{d}\Omega = \delta_{ij}$$

simplex elements: unique choice for momentum shape functions χ_i



displacement



linear momentum

TKACHUK, A., BISCHOFF, M. (2015).

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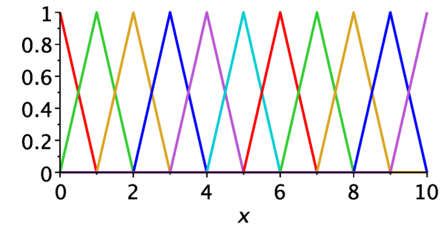
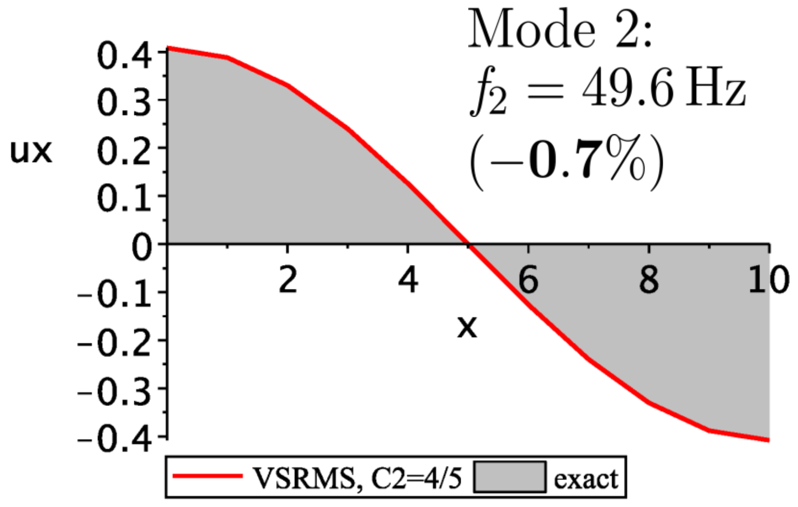
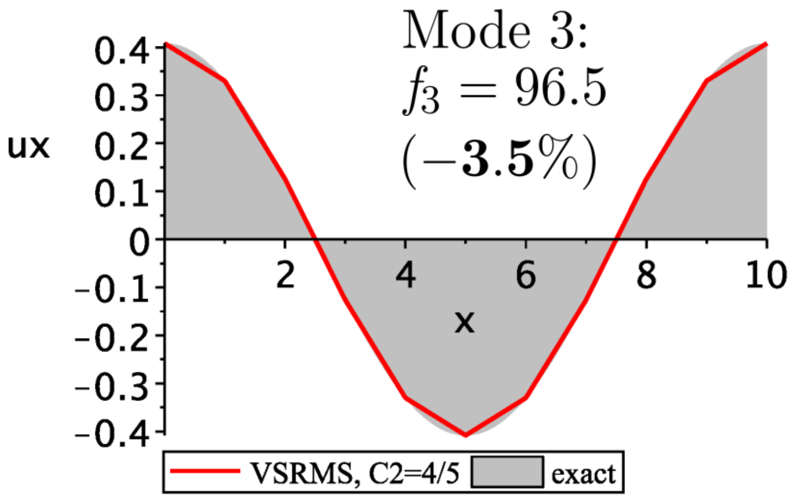
Uniform mesh, free-free rod

$E = 10^9$
$A = 1$
$\rho = 1000$
$l = 10$
$n = 10$



Accurate: ref. error for mode 2 and 3 with LMM are − 0.4% and − 1.6%

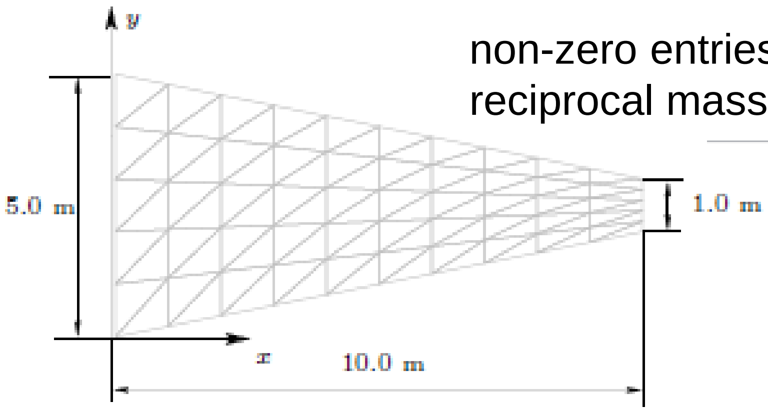


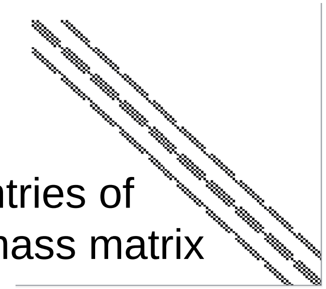


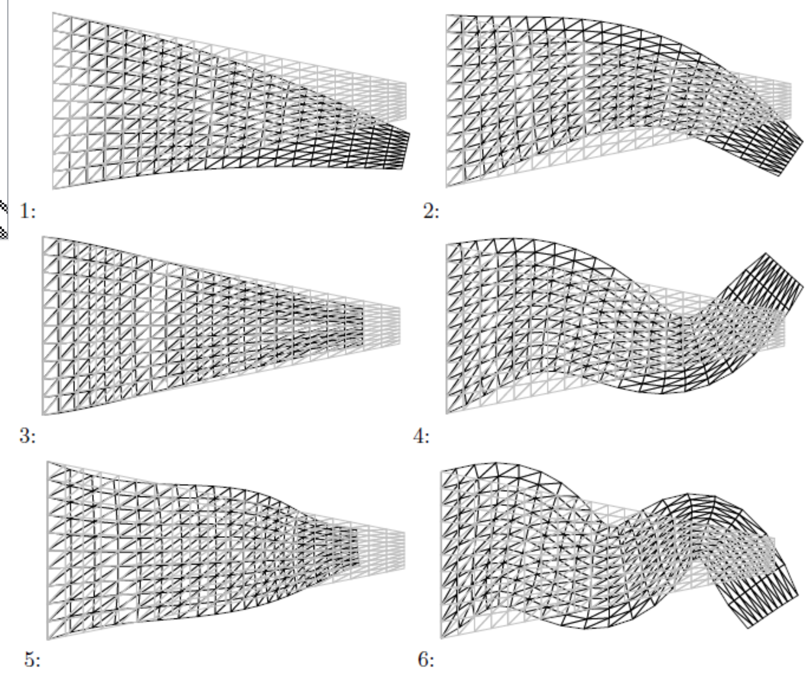
Efficient: the max. frequency is 77% of the value for LMM (+29% speed-up)

TKACHUK, A., BISCHOFF, M. (2015).

NAFEMS FV32 benchmark: computation of eigenvalues







$E = 200\,\text{GPa}$ Density $= 8000\,\text{kg/m}^3$
Poisson's ratio $= 0.3$ Plate thickness $= 0.05\,\text{m}$

TKACHUK, A., BISCHOFF, M. (2015).

	f_1, Hz	f_2, Hz	f_3, Hz	f_4, Hz	f_5, Hz	f_6, Hz	$f_{\max}$, Hz	$f^{\circ}_{\max}/f^{LMM}_{\max}$
Reference	44.623	130.03	162.70	246.05	379.90	391.44	-	-
LMM	45.421	132.60	162.73	251.40	387.40	391.19	18409.14	1.00
VSRMS, $C_2 = 0.99$	45.308	131.20	162.54	245.59	371.54	388.38	9221.93	0.50

Satisfactory results for lower frequencies and modes
Reduction of the highest frequency by 50%

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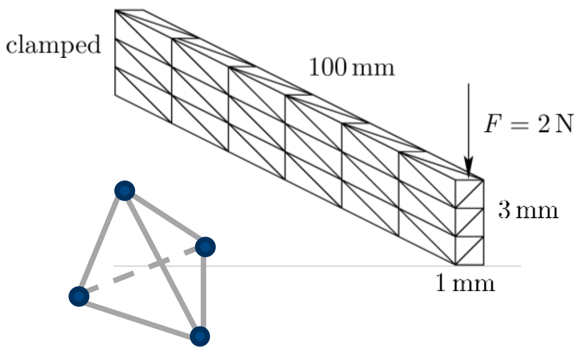
transient analysis of a cantilever beam

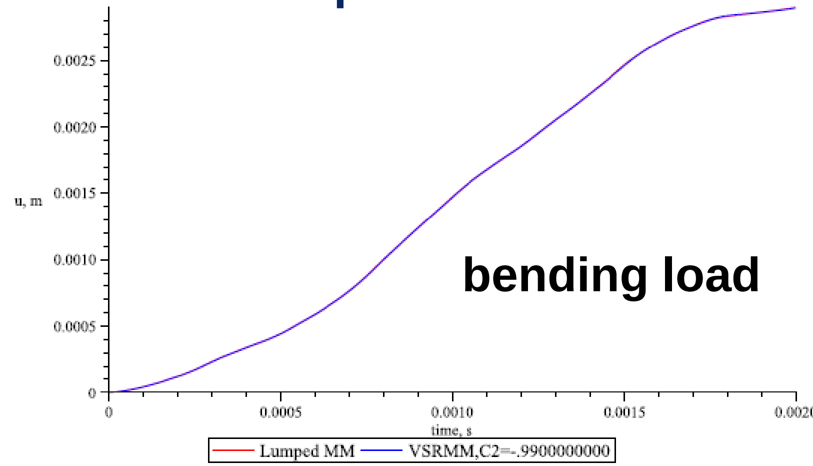


Tet4

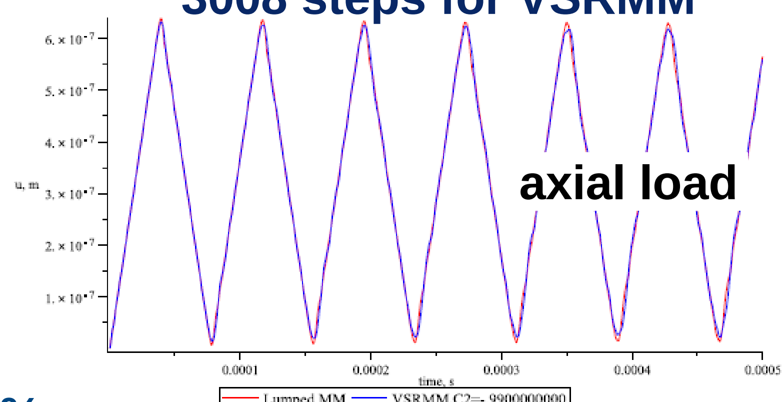
$$E = 2.07 \cdot 10^{11}\,\mathrm{N/m^2}$$
$$\rho = 7850\,\mathrm{kg/m^3}$$
$$\nu = 0.0$$
$$n_\mathrm{x} = 50$$
$$n_\mathrm{y} = 1$$
$$n_\mathrm{z} = 3$$
$$F = 2\,\mathrm{N}$$
$$t_\mathrm{end} = 20\,\mathrm{ms}$$

$$\Psi_{3D} = I_{3\times 3}$$

16000 steps for LMM vs.
12032 steps for VSRMM



4000 steps for LMM vs.
3008 steps for VSRMM



Very good results for bending load
Good results for axial load
Reduction of the highest frequency by 25%

CTU, Prague
05.06.2018

Lecture 8: Finite Element Method III
Mass matrices

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Appendix A: Recommended literature

Zienkiewicz, O. C., Taylor, R. L., & Zhu, J. Z. (2010). The Finite Element Method: Its Basis and Fundamentals, 6^{th} editions. (concise description of consistent and lumped mass matrices)

Belytschko, T., Liu, W. K., & Moran, B. *Nonlinear finite elements for continua and structures*. John Wiley, 2000. (also mass matrices for structural elements)

Cohen, G. C. *Higher-order numerical methods for transient wave equations.* Springer, 2003. (mass lumping for high-order hexahedral and tetrahedral elements)

Felippa, Carlos A. "Construction of customized mass-stiffness pairs using templates." *Journal of Aerospace Engineering* 19.4 (2006): 241-258. (good introduction to customization of mass matrices)

Mass Templates for Bar2 Elements
https://www.colorado.edu/engineering/CAS/courses.d/MFEMD.d/MFEMD.Ch22.d/MFEMD.Ch22.pdf