Lecture 8: Finite Element Method III
Mass matrices

Properties of mass matrix

Consistent and lumped mass matrix

High-order mass matrix

Direct inversion of mass matrix
Motivation for mass matrices

- discrete representation of inertia properties of structures
- needed for transient analysis \[ M\ddot{U} = F^{\text{ext}} - F^{\text{int}}(U) \]  
  Lectures 17 and 19
- modal and spectral analyses \( (K - \omega^2M)\phi = 0 \)  
  Lecture 16
- harmonic analysis \( (K + i\omega C - \omega^2M)U = F \)

- gravity load and Rayleigh model of damping  
  \[ F^{\text{grav}} = Mg \quad \quad C = a_k K + a_m M \]  
  Lecture 22

Mass matrices influence

- computational cost in explicit transient and modal analyses  
  Lectures 17 and 19
- accuracy through discretization (dispersion) error  
  Lectures 8 and 16

Understanding and choosing mass matrices is important!
Properties of mass matrix

Physical properties

- matrix symmetry
- positive definiteness (semi-positive)
- preservation of total translational mass
- physical symmetry

\[
M = M^T
\]
\[
v^T M v \geq 0 \quad v \neq 0
\]
\[
\int_\Omega \rho \, dV = 1_x^T M \, 1_x
\]

with unit motion in x-, y- or z-direction

\[
1_x = [1, 0, 0, 1, 0, 0, 1, 0, 0, \ldots]
\]

Algorithmic properties

- low fill-in or sparsity (ideally diagonal)
- easiness of computation
Consistent mass matrix for continua

**Virtual work principle**

\[ \int_{\Omega} \left( \delta \mathbf{u}^T \rho \ddot{\mathbf{u}} + \delta \varepsilon^T \sigma - \delta \mathbf{u}^T \mathbf{b} \right) \, dV - \int_{\Gamma_\sigma} \delta \mathbf{u}^T \dot{\mathbf{t}} \, dA = 0 \]

Spatial discretization

\[ \mathbf{u} \approx \mathbf{u}^h = \mathbf{N} \mathbf{U} \quad \delta \mathbf{u} \approx \delta \mathbf{u}^h = \mathbf{N} \delta \mathbf{U} \quad \varepsilon \approx \varepsilon^h = \mathbf{B} \mathbf{U} \quad \delta \varepsilon \approx \delta \varepsilon^h = \mathbf{B} \delta \mathbf{U} \]

Dynamic equation of motion

\[ \mathbf{M} \ddot{\mathbf{U}} = \mathbf{F}^{\text{ext}} - \mathbf{F}^{\text{int}} (\mathbf{U}) \]

Consistent mass matrix (CMM)

\[ \mathbf{M} = \int_{\Omega} \rho \mathbf{N}^T \mathbf{N} \, dV \]

Internal and external force vectors

\[ \mathbf{F}^{\text{int}} = \int_{\Omega} \mathbf{B}^T \sigma \, dV \]

\[ \mathbf{F}^{\text{ext}} = \int_{\Omega} \mathbf{N}^T \mathbf{b} \, dV \int_{\Gamma_\sigma} \mathbf{N}^T \dot{\mathbf{t}} \, dA \]
Consistent mass matrix for continua

**Example: 2-node truss in 1D**

Shape function in isoparametric coordinates

\[ N_1 = \frac{1}{2} (1 - \xi) \]
\[ N_2 = \frac{1}{2} (1 + \xi) \]

Consistent mass matrix on element level (constant density and cross-section)

\[ \mathbf{m}_e = \int_0^{l_e} \rho \mathbf{N}^T \mathbf{N} \, dx \]

\[ = \int_{-1}^{1} \frac{\rho l_e}{2} \begin{bmatrix} N_1 \cdot N_1 & N_1 \cdot N_2 \\ N_2 \cdot N_1 & N_2 \cdot N_2 \end{bmatrix} d\xi = \frac{\rho l_e A e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \]

\[ \mathbf{m}_e = \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix} \]

**Example: 3-node triangle in 2D**

Shape function in isoparametric coordinates

\[ N_1 = 1 - \xi - \eta \quad N_2 = \xi \quad N_3 = \eta \]

Consistent mass matrix on element level (constant density)

\[ \mathbf{m}_e = \int_A \rho \mathbf{N}^T \mathbf{N} t \, dA \]

\[ = \frac{\rho A t}{12} \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix} \]
Consistent mass matrix for continua

**Example: 4-node quadrilateral in 2D**

Shape function in isoparametric coordinates

\[ N_i = \frac{1}{2} (1 + \xi_i \xi)(1 + \eta_i \eta) \]

![4-node quadrilateral diagram]

Consistent mass matrix on element level via 2x2 numerical quadrature (Gauss)

\[
\mathbf{m}_e = \int_{\Omega_e} \rho \mathbf{N}^T \mathbf{N} t \, dx \, dy = \int_{-1}^{1} \int_{-1}^{1} \rho \mathbf{N}^T \mathbf{N} \left| J \right| t \, d\xi \, d\eta
\]

\[
\approx \sum_{GP=1}^{4} \left[ (\rho \mathbf{N}^T \mathbf{N} \left| J \right| t) \bigg|_{\xi = \xi_{GP}} \bigg|_{\eta = \eta_{GP}} w_{GP} \right]
\]

Note: 8-node hexahedron requires 3x3x3 quadrature

**Properties of consistent mass matrix**

- matrix symmetry (by construction)
  
  \[
  \mathbf{M} = \mathbf{M}^T
  \]

- positive definiteness (for full integration)

  \[
  \mathbf{v}^T \mathbf{M} \mathbf{v} \geq 0 \quad \mathbf{v} \neq 0
  \]

- preservation of total translational mass (for full integration)

  \[
  \int_{\Omega} \rho \, dV = \mathbf{1}_x^T \mathbf{M} \mathbf{1}_x
  \]

- physical symmetry

- low fill-in but non-diagonal

- easiness of computation
Consistent mass matrix for continua

Example: 10-node tetrahedral in 3D

Tensor products rule are not symmetric and leading to physically unsymmetric matrix. No closed expression of the quadrature are available → numerical search, e.g. Felippa rules

Available as FORTRAN and MATLAB code for 1-, 4-, 8-, 14-, 15-, 24-point r from https://people.sc.fsu.edu/~jburkardt/m_src/tetrahedron_felippa_rule/tetrahedron_felippa_rule.html
### Lumped mass matrix

**Motivation for lumped (diagonalized) mass matrices**

- Trivial computation of acceleration from the force vector:  \[ \ddot{U} = M^{-1}(F^{\text{ext}} - F^{\text{int}}(U)) \]
- Increased critical time step with respect to consistent mass matrix
- Reduced storage in RAM
- Reduced cost in modal analysis

**Methods for mass lumping (diagonalization)**

- Row-sum-lumping
- Nodal quadrature for elements with Gauss-Lobatto nodal location

\[ (K - \omega^2 M)\phi = 0 \]
Lumped mass matrix

Row-sum-lumping
1. Starting point: consistent mass matrix
2. Add all terms at each row to diagonal

\[ s_i = \sum_{j=1}^{n} m_{e,ij} \]

\[ m_e^D = \text{diag}(s_i) \]

Example: 2-node truss in 1D
Consistent mass matrix on element level (constant density and cross-section)

\[ m_e = \frac{\rho A l_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \]

Lumped mass matrix by row-sum-diagonalization

\[ m_e^D = \frac{\rho A l_e}{6} \begin{bmatrix} 2 + 1 & 0 \\ 0 & 1 + 2 \end{bmatrix} = \frac{\rho A l_e}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

Example: 3-node triangle in 2D
Consistent mass matrix on element level (constant density)

\[ m_e = \frac{\rho A t}{12} \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix} \]

Lumped mass matrix by row-sum-diagonalization

\[ m_e^D = \frac{\rho A t}{3} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]
Example: 8-node quadrilateral in 2D (Serendipity)

Shape function in isoparametric coordinates

\[
\begin{align*}
N_1 &= -0.25(1 - \xi)(1 - \eta)(1 + \xi + \eta) \\
N_2 &= -0.25(1 + \xi)(1 - \eta)(1 - \xi + \eta) \\
N_3 &= -0.25(1 + \xi)(1 + \eta)(1 - \xi - \eta) \\
N_4 &= -0.25(1 - \xi)(1 + \eta)(1 + \xi - \eta) \\
N_5 &= 0.5(1 - \xi^2)(1 - \eta) \\
N_6 &= 0.5(1 - \eta^2)(1 + \xi) \\
N_7 &= 0.5(1 - \xi^2)(1 + \eta) \\
N_8 &= 0.5(1 - \eta^2)(1 - \xi)
\end{align*}
\]

Lumped mass matrix by row-sum-diagonalization (constant density and Jacobian)

\[
m_e^D = \frac{\rho A t}{12} \text{diag}(-1, -1, -1, -1, 4, 4, 4, 4)
\]

negative terms at corner nodes!

The same problem is observed for quadratic 10-node tetrahedral finite element

Lumped mass matrix by row-sum-diagonalization (constant density and Jacobian)

\[
m_e^D = \frac{\rho V}{20} \text{diag}(-1, -1, -1, -1, 4, 4, 4, 4, 4, 4)
\]

negative terms at corner nodes!

Remedy: Hinton-Rock-Zienkiewicz method
Lumped mass matrix

Hinton-Rock-Zienkiewicz method

1. Starting point: consistent mass matrix
2. Compute some of all diagonal terms of CMM for one spatial direction
   \[ S = \sum_{i=1}^{n} m_{e,ii} \]
3. Compute diagonal terms as total mass of the elements scaled with ratio of diagonal term to the sum of diagonal terms
   \[ s_i = \frac{m_{e,ii}}{S} m_e \]
   \[ m_e^D = \text{diag}(s_i) \]

Example: 8-node quadrilateral in 2D (Serendipity)
Lumped mass matrix by HRZ lumping (constant density and Jacobian)
   \[ m_e^D = \frac{\rho A t}{36} \text{diag}(1,1,1,1,8,8,8,8) \]

Example: quadratic 10-node tetrahedral finite element
Lumped mass matrix by HRZ lumping (constant density and Jacobian)
   \[ m_e^D = \frac{\rho V}{108} \text{diag}(3,3,3,3,16,16,16,16,16,16,16,16) \]
High-order mass matrices

Motivation for high-order mass matrices

- the eigenfrequencies for CMM tend to be higher than analytical values
- the eigenfrequencies for LMM tend to be lower than analytical values
- weighted some of both can cancel the error and yield better results

Example: 2-node truss in 1D

Consistent and lumped mass matrices (constant density and cross-section)

\[
\mathbf{m}_e = \frac{\rho A l_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \mathbf{m}_e^D = \frac{\rho A l_e}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

Lumped mass matrix by row-sum-diagonalization

\[
\mathbf{m}_e^{HO} = 0.5 \mathbf{m}_e + 0.5 \mathbf{m}_e^D = \frac{\rho A l_e}{12} \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}
\]
**High-order mass matrices**

**Example: 2-node truss in 1D**

Shape function in isoparametric coordinates

\[ N_1 = -0.5(1 - \xi)\xi \]

\[ N_2 = 0.5(1 + \xi)\xi \]

\[ N_3 = 1 - \xi^2 \]

Consistent and lumped mass matrices (constant density and cross-section)

\[
\mathbf{m}_e = \frac{\rho A l_e}{30} \begin{bmatrix}
4 & -1 & 2 \\
-1 & 4 & 2 \\
2 & 2 & 16
\end{bmatrix}
\]

\[
\mathbf{m}_e^D = \frac{\rho A l_e}{6} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 4
\end{bmatrix}
\]

Lumped mass matrix by row-sum-diagonalization

\[
\mathbf{m}_{e}^{HO} = \frac{1}{3} \mathbf{m}_e + \frac{2}{3} \mathbf{m}_e^D = \frac{\rho A l_e}{90} \begin{bmatrix}
14 & -1 & 2 \\
-1 & 14 & 2 \\
2 & 2 & 56
\end{bmatrix}
\]
Reciprocal mass matrices

Motivation for reciprocal mass matrices

- trivial computation of acceleration from the force vector
- increased critical time step with respect to lumped mass matrix
- memory storage comparable with CMM
- more accurate than HRZ lumping for elements without stable row-sum-diagonalization

\[ \ddot{U} = C^o(F^{\text{ext}} - F^{\text{int}}(U)) \]

“directly” constructed inverse mass
Reciprocal mass matrices

Virtual work principle

\[
\int_{\Omega} \left( \delta \mathbf{u}^T \mathbf{\rho} \dot{\mathbf{u}} + \delta \mathbf{\varepsilon}^T \mathbf{\sigma} - \delta \mathbf{u}^T \mathbf{\hat{b}} \right) \, dV - \int_{\Gamma_\sigma} \delta \mathbf{u}^T \mathbf{\hat{t}} \, dA = 0
\]

“recast the term”

\[
\delta \mathbf{u}^T \mathbf{\dot{p}}
\]

\[
\delta \mathbf{u}^T \mathbf{\dot{p}} + \delta \mathbf{p}^T \left( \mathbf{\dot{u}} - \mathbf{\rho}^{-1} \mathbf{p} \right)
\]

\[
\delta \mathbf{u}^T \mathbf{\dot{p}} + \delta \mathbf{p}^T \left( \mathbf{\dot{u}} - (1 - C_2)\mathbf{\rho}^{-1} \mathbf{p} - C_2 \mathbf{v} \right) + \delta \mathbf{v}^T C_2 (\mathbf{\rho} \mathbf{\dot{v}} - \mathbf{p})
\]

\[
\mathbf{\dot{u}} = \mathbf{v} \quad \text{(velocity)}
\]

\[
C_2' \quad \text{(free parameter)}
\]

Reciprocal mass matrices

Parametrized virtual work principle

\[ \int_{\Omega} \left( \delta u^T \ddot{p} + \delta p^T \left( \ddot{u} - (1 - C_2)\rho^{-1} p - C_2 v \right) + \delta v^T C_2 (\rho \dot{v} - p) \right) \, dV \\
+ \int_{\Omega} \left( \delta \varepsilon^T \sigma - \delta u^T \dot{b} \right) \, dV - \int_{\Gamma_\sigma} \delta u^T \dot{t} \, dA = 0 \]

Spatial discretization

\[ v \approx v^h = \Psi V \quad \quad \quad p \approx p^h = \chi P \]

Dynamic equation of motion

\[ \begin{cases} 
A \ddot{P} = \mathbf{F}^{\text{ext}} - \mathbf{F}^{\text{int}} \\
A^T \dot{U} = (1 - C_2)CP + C_2 WV \\
YV = W^T P 
\end{cases} \]

Spatial discretization

\[ A = \int_{\Omega} N^T \chi^T d\Omega, \quad \text{projection} \quad \quad \quad C = \int_{\Omega} \rho^{-1} \chi^T \chi d\Omega, \quad \text{reciprocal mass matrix} \]

\[ Y = \int_{\Omega} \rho \Psi^T \Psi d\Omega, \quad \text{mass with } v \quad \quad \quad W = \int_{\Omega} \chi^T \Psi d\Omega \quad \text{projection} \]
Reciprocal mass matrices

\[
\begin{align*}
A \dot{P} &= F^{\text{ext}} - F^{\text{int}} \\
A^T \ddot{U} &= (1 - C^2) C P + C_2 W V \\
Y V &= W^T P
\end{align*}
\]

Elimination of velocity and momentum degrees of freedom by static condensation

\[
\begin{align*}
V &= Y^{-1} W^T P \\
\dot{U} &= C^0 P
\end{align*}
\]

biorthogonality condition

\[
A_{ij} = \int_\Omega N_i \chi_j \, d\Omega = \delta_{ij} \\
\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}
\]

\[
\dot{U} = C^0 (F^{\text{ext}} - F^{\text{int}})
\]

\[
C^0 = C + C_2 (W Y^{-1} W^T - C)
\]

Dynamic equation of motion with scaled reciprocal mass matrix

Positive definite:

\[
C^0 = C + \tilde{\lambda}^0 = (1 - C^2) \underbrace{C}_{(1 - C^2) C} + C_2 \underbrace{W Y^{-1} W^T}_{W^T W}
\]

\[
0 < C_2 < 1
\]

Sparse fill-in of CMM:

\[
Y
\]

is block diagonal for element-wise interpolation of \( v \)

Reciprocal mass matrices in explicit dynamics

**velocity shape functions** $\Psi$

1D: $\Psi = \begin{bmatrix} 1 \end{bmatrix}$

2D: $\Psi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Element-wise interpolation

**linear momentum shape functions** $\chi$

Biorthogonality condition

$$A_{ij} = \int_{\Omega} N_i \chi_j \, d\Omega = \delta_{ij}$$

**simplex elements:** unique choice for momentum shape functions $\chi_i$

Reciprocal mass matrices

Uniform mesh, free-free rod

\[ E = 10^9 \]
\[ A = 1 \]
\[ \rho = 1000 \]
\[ l = 10 \]
\[ n = 10 \]

Accurate: ref. error for mode 2 and 3 with LMM are − 0.4% and − 1.6%

Efficient: the max. frequency is 77% of the value for LMM (+29% speed-up)

Reciprocal mass matrices

NAFEMS FV32 benchmark: computation of eigenvalues

non-zero entries of reciprocal mass matrix

\[ E = 200 \text{ GPa} \quad \text{Density} = 8000 \text{ kg/m}^3 \]

Poisson’s ratio = 0.3  \quad \text{Plate thickness} = 0.05 \text{ m}


<table>
<thead>
<tr>
<th></th>
<th>( f_1 ), Hz</th>
<th>( f_2 ), Hz</th>
<th>( f_3 ), Hz</th>
<th>( f_4 ), Hz</th>
<th>( f_5 ), Hz</th>
<th>( f_6 ), Hz</th>
<th>( f_{\text{max}} ), Hz</th>
<th>( f_{\text{max}}^\circ / f_{\text{max}}^{\text{LMM}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reference</td>
<td>44.623</td>
<td>130.03</td>
<td>162.70</td>
<td>246.05</td>
<td>379.90</td>
<td>391.44</td>
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<td>-</td>
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<td>LMM</td>
<td>45.421</td>
<td>132.60</td>
<td>162.73</td>
<td>251.40</td>
<td>387.40</td>
<td>391.19</td>
<td>18409.14</td>
<td>1.00</td>
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<tr>
<td>VSRMS, ( C_2 = 0.99 )</td>
<td>45.308</td>
<td>131.20</td>
<td>162.54</td>
<td>245.59</td>
<td>371.54</td>
<td>388.38</td>
<td>9221.93</td>
<td>0.50</td>
</tr>
</tbody>
</table>

Satisfactory results for lower frequencies and modes
Reduction of the highest frequency by 50%
Reciprocal mass matrices

transient analysis of a cantilever beam

\[ E = 2.07 \cdot 10^{11} \text{ N/m}^2 \]
\[ \rho = 7850 \text{ kg/m}^3 \]
\[ \nu = 0.0 \]
\[ n_x = 50 \]
\[ n_y = 1 \]
\[ n_z = 3 \]
\[ F = 2 \text{ N} \]
\[ t_{\text{end}} = 20 \text{ ms} \]

16000 steps for LMM vs. 12032 steps for VSRMM

bending load

4000 steps for LMM vs. 3008 steps for VSRMM

axial load

Very good results for bending load
Good results for axial load
Reduction of the highest frequency by 25%
Lecture 8: Finite Element Method III
Mass matrices

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Appendix A: Recommended literature


Mass Templates for Bar2 Elements
https://www.colorado.edu/engineering/CAS/courses.d/MFEMD.d/MFEMD.Ch22.d/MFEMD.Ch22.pdf